

Exact solutions for classical few-body systems from the multichannel quantum
inverse problem

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Abstract

A surprising "duality" of the Newton equation with *time-dependent* forces and the *stationary* Schrödinger equation is discussed. Wide classes of exact solutions not known before for few-body Newton equations are generated directly from exactly solvable multichannel models discovered in quantum mechanics due to the inverse problem and the supersymmetry (SUSYQ) approach. The application of this duality to the control of the stability (bifurcations) of classical motions is suggested.

1 Introduction

In this paper the results related to the control of classical few-body systems are presented. We use the duality of corresponding solutions with the known exact solutions of quantum problems. The complete sets of such exactly solvable models have been recently intensively discussed [1, 2]. The transition between quantum and classical solutions is possible through simply renaming variables and functions.

Physics is permanently developing in a tight interplay with mathematics ("the laws of nature are written in the mathematical language"). There are cases when the same mathematical equation can describe physical systems of different nature, which broadens and deepens their understanding. We shall demonstrate this for the example of the multichannel formalism that is a powerful and universal tool in quantum inverse and direct problems of few-body physics [1, 2, 3].

We start with briefly reminding the one-dimensional and one-channel case.

The corresponding stationary Schrödinger equation

$$-\psi''(x) + v(x)\psi(x) = E\psi(x) \quad (1)$$

can be rewritten by changing the notation

$$\psi \rightarrow z; \quad x \rightarrow t; \quad d^2/dx^2 \rightarrow d^2/dt^2; \quad [v(x) - E]\psi \rightarrow F(t, z)$$

so that it will look like an equation of motion of a classical particle under the action of the force $F(t, z)$ dependent on time t and coordinate z :

$$\ddot{z}(t) = [v(t) - E] \quad z \equiv F(t, z). \quad (2)$$

It is a Newton equation: the acceleration of a particle moving along the z axis is determined by the force $F(t, z)$ linearly dependent on z (an oscillator-type potential $\sim z^2$) with a time-dependent strength. In a classical equation, the former quantum energy E and the potential $v(x)$ become, respectively, a parameter and a function characterizing the external force.

This information was obtained at the School on Quantum Mechanics in Mexico, 1998, from Prof. Bogdan Mielnik and Dr. David Fernandez and paper [4], see also references therein. It was surprising for us and particularly interesting because it means that complete sets of exactly solvable quantum models [2, 5] simultaneously provide exact solutions to classical problems of quite a different nature.

It is also a "paradox" that the eigenvalues E_n for *stable* bound states in quantum mechanics correspond to the points of bifurcations (absolute instability) of solutions for classical motion. Let us comment on this statement. At an energy slightly lower than a ground state energy level E_n , the nonphysical solution of the Schrödinger equation decreasing asymptotically to the left grows exponentially to the right. And somewhat above the energy level, the solution with the same asymptotic behavior on the left bends more strongly, acquires an additional node and grows exponentially to the right with another sign. In the classical case it corresponds to an instantaneous change of the regime for a particle motion when the parameter E goes through the point E_n : a particle being initially at the origin in the first case goes to the right with increasing accelerations and in the second case it goes to the left. At the point $E = E_n$ the particle localized previously at the origin after the influence of the time-depending external field returns to the origin.

So, the possibility of shifting, creating and removing arbitrary energy levels in quantum systems [1, 2, 5] corresponds in the classical case to the ability to control the bifurcation points which are the parameters of principal importance characterizing the instability of motion.

Of course, at the same value E , a classical scattering with another initial ($t \rightarrow -\infty$) condition is possible corresponding to another dual quantum asymptotic ($x \rightarrow -\infty$) condition.

In the quantum inverse problem, the potential is uniquely determined by a complete set of spectral parameters : energy level values, normalizing parameters (spectral weights) and phase shifts (or resonance positions if S-matrix has fractional-rational form [7]) for scattering. Therefore, classical counterparts of these parameters (e.g. bifurcations) are control levers of the corresponding classical system. This interesting fact was not so evident from the point of view of Newton equations.

The intuition developed in exactly solvable quantum models [1, 5] allows one to qualitatively predict also classical motion without computation.

2 Multichannel, multi-dimensional and few-body systems

The multichannel systems [6, 7, 8, 9, 10] for vector-valued wave functions with partial channel components $\psi_i(x)$ are simply matrix generalizations of the one-dimensional Schrödinger equation (1)

$$-\psi_i''(x) + \sum_j v_{ij}(x)\psi_j(x) = E_i\psi_i(x),$$

$$E_i \equiv E - \epsilon_i \tag{3}$$

where the interaction matrix $v_{ij}(x)$ replaces the ordinary potential and ϵ_i are threshold energies that can be different in partial channels. This system can be rewritten as multi-dimensional or few-body equations of classical mechanics with special forces.

Let us replace the partial channel wave function $\psi_i(x)$ and its space coordinate variable x in the system (3) by the coordinate z_i of the i th classical particle and time: $z_i(t)$. Then the second derivative $\psi_i(x)$ is substituted by the acceleration \ddot{z}_i and all other terms can be considered as the forces $F_i(t, z)$ dependent on time and space coordinates acting on the i th particle. So, elements of the potential matrix $v_{ij}(x)$ and channel energy terms $E_i\psi_i(x)$ become constituents of the forces

$$F_{ij}(t) = \sum_j v_{ij}(t)z_j(t) - E_iz_i(t).$$

The whole system (3) becomes a system of equations for several classical particles, where the particle accelerations are determined by functions of the time-dependent forces

$$\ddot{z}_i = \sum_j v_{ij}(t)z_j(t) - E_iz_i(t) \equiv F_i(t). \quad (4)$$

Again as in (2) the parameters E_i are no longer the channel energies, but simply enter as parameters into the definition of the forces acting in the system. Therefore, every exact solution of the multichannel quantum problem (and complete sets of them) corresponds to an exact solution of the few-body classical problem with time-dependent forces. *The asymptotic and boundary conditions of quantum problems determine initial and final conditions of the corresponding classical solutions.* For the classical scattering solutions it is possible to use nonphysical quantum solutions growing *linearly* in asymptotic regions: $\psi(x) = ax + b$ (for $E_i = 0$). The constants a and b fixing the inclination of the straight line and the position point of the node of the wave function determine in the classical problem the position and velocity of a particle in some moment of time (for instance, $t=0$): $z = at + b$.

As an example of the prediction of a peculiar behavior of few-body systems, one can choose the dual model of multichannel exact solution when one of the partial channel wave functions is concentrated at the origin. This corresponds to increasing the spectral weight parameter of this partial channel $C_i \equiv \frac{d\psi_i(x)}{dx}|_{x=0}$. We have discovered [5] that in this case all other partial components of wave functions are transferred to the chosen i th channel. In the limiting case $C_i \rightarrow \infty$ the whole wave function is totally gathered in the i th channel, so that all other channels are "emptied". In the classical interpretation, it means that it is easy to determine the forces $F_i(t)$ which move mainly only one chosen i th particle. Although these forces are strong enough to expect that all other particles are not passive spectators of intensive motion of their chosen companion.

To give an idea of the character of explicit formulae of exactly solvable multichannel models arising from the inverse problem, we show an example of the reflectionless interaction matrix with one bound state at energy $E_{bound} < \epsilon_i$:

$$v_{ij}(x) = 2 \frac{d}{dx} \frac{M_i M_j e^{-(\kappa_i + \kappa_j)x}}{1 + \sum_m \frac{M_m^2}{2\kappa_m} e^{-2\kappa_m x}}, \quad (5)$$

$$\kappa_i = \sqrt{\epsilon_i - E_{bound}},$$

where M_i stand for partial channel spectral parameters which are pre-exponential factors in the decreasing asymptotic tails of the bound state wave functions. An interested reader can find other formulae for a complete set of inverse problem and SUSYQ transformations in [2].

The transformation of the three-channel system of type (3) by renaming:

$$\psi_1(x) \rightarrow x(t), \quad \psi_2 \rightarrow y(t), \quad \psi_3 \rightarrow z(t)$$

gives us the classical equations for the three-dimensional motion of a particle.

As interesting instructive examples can be considered the classical counterparts of quantum systems with "paradoxical" coexistence of bound and scattering states at the same energy [5], 1999 or combinations of absolute transparency and strong reflection for different linearly independent solutions of multichannel equations.

It is possible to significantly extend the class of exactly solvable models if we introduce, into the quantum equations nonhomogeneous terms which can be treated as sources with an arbitrary dependence on coordinates. From the classical point of view, they correspond to forces $F_n(t)$ which are independent of coordinates. Having the sets of independent exact solutions of the homogeneous equations we get the expressions for Green functions. They give explicit analytic expressions for the solutions of the initial nonhomogeneous equations.

The quantum multichannel systems obtained by adiabatic expansions have derivatives of first order which can be interpreted as friction forces depending on the velocity. But exact solutions of these systems remain an open problem yet.

3 Conclusion

The authors consider it a pleasure and an honor to participate in this issue of the Few-Body Systems dedicated to Prof. W. Glöckle, the scientist having record results in this fundamental field of quantum physics. Already his first work [11] produced a strong impression on us. It was a significant contribution to the multichannel formalism. Prof. W. Glöckle also stimulated our investigations on qualitative theory of quantum design reviewed later in the books "Lessons on Quantum Intuition", ¹ "New ABC of Quantum Mechanics (in pictures)" ² and in a series of review articles [1, 2]. We appreciate very much that Prof. W. Glöckle was one of the first physicists in the Western Europe who organized, at Ruhr University at his seminar consideration of our theory.

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¹ With the subtitles: Gallery of wonderful potentials. Algorithms of spectral, scattering and decay control. Exactly solvable models of inverse problems and SUSYQ. Universal building elements "bricks and blocks" of quantum systems.

² With the subtitles: Qualitative foundations of the wave literacy. Construction of quantum systems with the given properties.

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